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# Laplace–Mellin transform of the non-holomorphic Eisenstein series

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## 1 Introduction and statement of the results

In this report, we give a Laplace–Mellin transform of the non-holomorphic Eisenstein series and the Fourier coefficients of the Eisenstein series respectively. The main theorem appeared in author's previous paper [5] (2007). We give a new proof by means of Mellin–Barnes type integral formulas in section 4.

Let  $k \geq 0$  be an even integer, Let  $i$  be the imaginary unit,  $s$  be a complex number whose real part  $\sigma$  and imaginary part  $t$ . As usual,  $H$  is the upper half plane. The non-holomorphic Eisenstein series for  $SL_2(\mathbb{Z})$  is defined by

$$E_k(z, s) = y^s \sum_{\{c, d\}} (cz + d)^{-k} |cz + d|^{-2s}. \quad (1)$$

Here  $z$  is a point of  $H$ ,  $s$  is a complex variable and the summation is taken over  $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ , a complete system of representation of  $\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbb{Z})\right\} \setminus SL_2(\mathbb{Z})$ . The right-hand side of (1) converges absolutely and locally uniformly on  $\{(z, s) \mid z \in H, \operatorname{Re}(s) > 1 - \frac{k}{2}\}$ , and  $E_k(z, s)$  has a meromorphic continuation to the whole  $s$ -plane.

Our main result is as follows:

**Theorem 1** ([5] Theorem 1) (I) Assume that  $k \geq 4$ . Then the Eisenstein series  $E_k(z, s)$  is a  $C^\infty$ -modular form of weight  $k$ , and of bounded growth for  $2 - k < \operatorname{Re}(s) < -1$  except on the poles. Further, for  $1 - \frac{k}{2} < \operatorname{Re}(s) < -1$  and  $n \in \mathbb{Z}_{>0}$ ,

$$\int_0^1 \int_0^\infty E_k(z, s) \exp(-2\pi i n \bar{z}) y^{k-2} dy dx = 0. \quad (2)$$

(II) Let  $E_k(z, s) = \sum_{n=-\infty}^\infty a(n; y, s) e^{2\pi i n x}$  be the Fourier expansion of the Eisenstein series. Then Fourier coefficients of the projection of  $E_k(z, s)$  to the space of holomorphic cusp forms are zero, namely,

$$(2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n; y, s) e^{-2\pi n y} y^{k-2} dy = 0 \quad (3)$$

for  $1 - k < \operatorname{Re}(s) < 0$  and  $n \in \mathbb{Z}_{>0}$ .

## 2 Projection to the space of cusp forms

The  $C^\infty$ -automorphic forms of bounded growth are introduced by Sturm in the study of zeta-functions of Rankin type.

The function  $F$  is called a  $C^\infty$ -modular form of weight  $k$ , if  $F$  satisfies the following conditions:

(A.1)  $F$  is a  $C^\infty$ -function from  $H$  to  $\mathbb{C}$ ,

(A.2)  $F((az+b)(cz+d)^{-1}) = (cz+d)^k F(z)$  for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ .

We denote by  $\mathfrak{M}_k$  the set of all  $C^\infty$ -modular forms of weight  $k$ . The function  $F \in \mathfrak{M}_k$  is called of bounded growth if for every  $\varepsilon > 0$

$$\int_0^1 \int_0^\infty |F(z)| y^{k-2} e^{-\varepsilon y} dy dx < \infty.$$

Let  $k$  be a positive even integer and  $S_k$  be the space of cusp forms of weight  $k$  on  $SL_2(\mathbb{Z})$ . For  $F \in \mathfrak{M}_k$  and  $f \in S_k$ , we define the Petersson inner product as usual

$$(f, F) = \int_{SL_2(\mathbb{Z}) \backslash H} f(z) \overline{F(z)} y^{k-2} dx dy.$$

In 1981, Sturm [7] constructed a certain kernel function by using Poincaré series, and showed the following theorem:

**Theorem 2** (Sturm 1981) *Assume that  $k > 2$ . Let  $F \in \mathfrak{M}_k$  be of bounded growth with the Fourier expansion  $F(z) = \sum_{n=-\infty}^{\infty} a(n, y) e^{2\pi i n x}$ . Let*

$$c(n) = (2\pi n)^{k-1} \Gamma(k-1)^{-1} \int_0^\infty a(n, y) e^{-2\pi n y} y^{k-2} dy.$$

*Then  $h(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \in S_k$  and*

$$(g, F) = (g, h)$$

*for all  $g \in S_k$ .*

## 3 Fourier expansion of the Eisenstein series

Next, we recall the Fourier expansion and the growth condition of  $E_k(z, s)$ . Let  $e(u) := \exp(2\pi i u)$  for  $u \in \mathbb{C}$ . For  $z \in H$  and  $\text{Re}(s) > 1 - \frac{k}{2}$ ,  $E_k(z, s)$  has an expansion:

$$E_k(z, s) = y^s + a_0(s) y^{1-k-s} + \frac{y^s}{\zeta(k+2s)} \sum_{m \neq 0} \sigma_{1-k-2s}(m) a_m(y, s) e(mx), \quad (4)$$

where

$$\begin{aligned} a_0(s) &= (-1)^{\frac{k}{2}} 2\pi \cdot 2^{1-k-2s} \frac{\zeta(k+2s-1) \Gamma(k+2s-1)}{\zeta(k+2s) \Gamma(s) \Gamma(k+s)}, \\ \sigma_s(m) &= \sum_{d|m, d>0} d^s, \end{aligned}$$

and

$$a_m(y, s) = \int_{-\infty}^{\infty} e(-mu) (u+iy)^{-k} |u+iy|^{-2s} du. \quad (5)$$

We call the first two terms of (4) are the constant term of  $E(z, s)$ . The integral (5) is entire function in  $s$  and of exponential decay in  $y|m|$ . This fact gives the meromorphical continuation of  $E_k(z, s)$  to the whole  $s$ -plane, and shows that the constant terms represent the  $y$ -aspect of  $E(z, s)$  when  $y$  tends to  $\infty$ . Namely, there exist positive constants  $A_1$  and  $A_2$  depending only on  $k$  and  $s$  such that

$$|E_k(z, s)| \leq A_1 y^{\operatorname{Re}(s)} + A_2 y^{1-\operatorname{Re}(s)-k} \quad (y \rightarrow \infty), \quad (6)$$

except on the poles. Further, the integral (5) is expressed in terms of special functions:

$$a_m(y, s) = \begin{cases} \frac{(-1)^{\frac{k}{2}} (2\pi)^{k+2s} m^{k+2s-1}}{\Gamma(k+s)} e^{-2\pi y m} \Psi(s, k+2s; 4\pi y m) & (m > 0), \\ \frac{(-1)^{\frac{k}{2}} (2\pi)^{k+2s} |m|^{k+2s-1}}{\Gamma(s)} e^{-2\pi y |m|} \Psi(k+s, k+2s; 4\pi y |m|) & (m < 0). \end{cases} \quad (7)$$

Here  $\Psi(\alpha, \beta; z)$  is the confluent hypergeometric function defined for  $\operatorname{Re}(z) > 0$  and  $\operatorname{Re}(\alpha) > 0$  by the following

$$\Psi(\alpha, \beta; z) := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-zu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} du,$$

(See for example [4] §7.2.) Then we have

**Proposition 1** *Assume  $E_k(z, s)$  is holomorphic at  $s \in \mathbb{C}$ . Then, there exist positive constants  $A_1$  and  $A_2$  depending only on  $k$  and  $s$  such that*

$$|E_k(x+iy, s)| \leq \begin{cases} A_1 (y^{-\operatorname{Re}(s)-k} + y^{\operatorname{Re}(s)}) & (\operatorname{Re}(s) > \frac{1-k}{2}) \\ A_2 (y^{-1+\operatorname{Re}(s)} + y^{1-\operatorname{Re}(s)-k}) & (\operatorname{Re}(s) \leq \frac{1-k}{2}) \end{cases}$$

for every  $y > 0$ .

**Remark 1.** The integral (5) plays a fundamental role in the study of automorphic forms. The initial work is due to Hecke [2]. His approach is explicated in

Schoenberg's book [6] (pp. 63-68). The representation (7) was originally investigated by Maass [3] (pp. 209-211). He used the Whittaker function to express the integral (7).

**Remark 2.** The estimation (6) is well-known, and Proposition 1 is the consequence of (6) and modularity of  $y^{\frac{k}{2}}E_k(z, s)$ .

## 4 Proof of Theorem 1

The orthogonality of  $E_k(z, s)$  and cusp forms gives the equation (2) under the convergence of the integral. Here Proposition 1 is used, however, we omit the detail of the proof of Theorem 1 (I) (see [5]). In this section, we give a new proof of Theorem 1 (II) by using the **Mellin-Barnes integral**. The following proposition is crucial to evaluate the integral in (3). (See [1], Section 6.5.)

**Proposition 2** (*Mellin-Barnes integral*)  $\Psi(\alpha, \beta; z)$  has an integral representation

$$\Psi(\alpha, \beta; z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha + w)\Gamma(-w)\Gamma(1 - \beta - w)}{\Gamma(\alpha)\Gamma(\alpha - \beta + 1)} z^w dw,$$

where  $|\arg(z)| \leq \frac{3}{2}\pi$ ,  $\alpha \notin \mathbb{Z}_{\leq 0}$ ,  $\alpha - \beta + 1 \notin \mathbb{Z}_{\leq 0}$  and the path of integration is indented so as to separate the poles of  $\Gamma(\alpha + w)$  and  $\Gamma(-w)\Gamma(1 - \beta - w)$ .

**Proof of Theorem 1 (II)** By Proposition 2, we have

$$\begin{aligned} & \int_0^\infty \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy \\ &= \frac{1}{2\pi i \Gamma(s)\Gamma(1 - k - s)} \int_0^\infty \int_L \Gamma(s + w)\Gamma(-w)\Gamma(1 - k - 2s - w) \\ & \quad \times (4\pi n)^w e^{-4\pi ny} y^{w+k+s-2} dw dy \end{aligned}$$

for  $s \notin \mathbb{Z}$ . Here the path of integration in  $w$  is denoted by  $L$  taken from  $-i\infty$  to  $i\infty$  so as to separate the poles of  $\Gamma(s + w)$  and  $\Gamma(-w)\Gamma(1 - k - 2s - w)$ . Then the interchange of the order of integration (11) is justified by Fubini's theorem in the region  $1 - k < \operatorname{Re}(s) < 0$ .

We also employ Barnes' lemma:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\alpha + w)\Gamma(\beta + w)\Gamma(\gamma - w)\Gamma(\delta - w) dw = \frac{\Gamma(\alpha + \gamma)\Gamma(\alpha + \delta)\Gamma(\beta + \gamma)\Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)},$$

where the path of integration is taken so as to separate the poles of  $\Gamma(\alpha + w)\Gamma(\beta + w)$  and  $\Gamma(\gamma - w)\Gamma(\delta - w)$ . (See [8] Section 14.52.) By the above lemma, we have

$$\begin{aligned} & \int_0^{\infty} \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy \\ &= \frac{(4\pi n)^{1-k-s}}{2\pi i} \Gamma(-s) \Gamma(-1 + k + s) \cdot \Gamma(0)^{-1} \\ &= 0, \end{aligned}$$

for  $1 - k < \operatorname{Re}(s) < 0$  and  $s \notin \mathbb{Z}$ .

If  $\alpha$  is zero or a negative integer,  $\Psi(\alpha, \beta; z)$  is a polynomial in  $z$ , which is called the (generalized) **Laguerre polynomial**. The Laguerre polynomials  $L_n^a(x)$  for  $n \in \mathbb{Z}_{\geq 0}$  are defined by

$$L_n^a(x) = \sum_{m=0}^n \binom{n+a}{n-m} \frac{(-x)^m}{m!} = \frac{(-1)^n}{n!} \Psi(-n, a+1; x),$$

which are known as the orthogonal polynomials associated with the scalar product

$$(\varphi_1, \varphi_2) := \int_0^{\infty} \varphi_1(x) \varphi_2(x) e^{-x} x^a dx$$

for  $a > -1$ . Especially, for positive integer  $n$ ,

$$(L_n^a, L_0^a) = \int_0^{\infty} L_n^a(x) e^{-x} x^a dx = 0. \quad (8)$$

By (11), we have for  $s \in \mathbb{Z}_{\leq 0}$ ,

$$\begin{aligned} & \int_0^{\infty} \Psi(s, k + 2s; 4\pi ny) e^{-4\pi ny} y^{k+s-2} dy \\ &= (-s)! (-1)^s \int_0^{\infty} L_{-s}^{k+2s-1}(4\pi ny) e^{-4\pi ny} y^{k+s-2} dy. \end{aligned} \quad (9)$$

Applying the relation  $L_n^{a-1}(x) = L_n^a(x) - L_{n-1}^a(x)$  for  $(-s-1)$ -times, we resolve (10) into (9) and complete the proof of Theorem 1 (II).  $\square$

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